# Supplementary Material of Handling Slice Permutations Variability in Tensor Recovery

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**Property 7.** *(Zhang 2017) If $P \in \mathbb{R}^{n\times n}$ is a permutation matrix, then $P^TP = PP^T = I$.*

**Property 1.** *For $A \in \mathbb{R}^{n_1\times n_2}$, then nuclear norm satisfy row (or column) permutations invariance,* i.e. *$\|PA\|_* = \|A\|_*$ for any permutation matrix $P \in \mathbb{R}^{n_1\times n_1}$ (or $\|AP\|_* = \|A\|_*$ for any permutation matrix $P \in \mathbb{R}^{n_2\times n_2}$).*

*Proof.* $\|PA\|_* = \|A\|_*$ by Property 7 and the unitary invariant norm property.

Similarly, we can get $\|AP\|_* = \|A\|_*$ for any permutation matrix $P \in \mathbb{R}^{n_2\times n_2}$. □

**Theorem 2.** *For $Y \in \mathbb{R}^{n_1\times n_2}$, $\mathcal{D}_\tau(Y) = P^{-1}\mathcal{D}_\tau(PY)$ for any permutation matrix $P \in \mathbb{R}^{n_1\times n_1}$ (and $\mathcal{D}_\tau(Y) = \mathcal{D}_\tau(YP)P^{-1}$ for any permutation matrix $P \in \mathbb{R}^{n_2\times n_2}$), where $\mathcal{D}_\tau(Y) = \arg\min_X \frac{1}{2}\|Y - X\|_F^2 + \tau\|X\|_*$, and $P^{-1}$ is an inverse operator of $P$.*

*Proof.*

$$\begin{aligned} P^{-1}\mathcal{D}_\tau(PY) &= P^{-1}\arg\min_Z \frac{1}{2}\|PY - Z\|_F^2 + \tau\|Z\|_* \\ &= \arg\min_X \frac{1}{2}\|PY - PX\|_F^2 + \tau\|PX\|_* \\ &= \arg\min_X \frac{1}{2}\|Y - X\|_F^2 + \tau\|X\|_*, \end{aligned} \tag{1}$$

where the second equation holds by letting $X = P^{-1}Z$, and the third equation holds by the Property 7 and Property 1.

Similarly, we can get $\mathcal{D}_\tau(Y) = \mathcal{D}_\tau(YP)P^{-1}$ for any permutation matrix $P \in \mathbb{R}^{n_2\times n_2}$. □

**Property 2.** *For $\mathcal{A} \in \mathbb{R}^{n_1\times n_2\times n_3}$, then $\sum_{i=1}^3 \alpha_i\|(\mathcal{A} \circ \mathcal{P}^{(k)})_{(i)}\|_* = \sum_{i=1}^3 \alpha_i\|\mathcal{A}_{(i)}\|_*$ for any slice permutations $\mathcal{P}^{(k)}$* i.e. $(k = 1, 2, 3)$, *where $\mathcal{A}_{(i)}$ represents the mode-i unfolding matrix of $\mathcal{A}$, $(\mathcal{A} \circ \mathcal{P}^{(k)})(k = 1, 2, 3)$ stands for the result by perform horizontal slice permutations, lateral slice permutations and frontal slice permutations on $\mathcal{A}$, respectively.*

*Proof.* For any slice permutations $\mathcal{P}^{(k)}(k = 1, 2, 3)$, exist permutation marries $P_i$ and $Q_i$ makes $(\mathcal{A} \circ \mathcal{P}^{(k)})_{(i)} = P_i\mathcal{A}_{(i)}Q_i$ for $i = 1, 2, 3$. Therefore, $\sum_{i=1}^3 \alpha_i\|(\mathcal{A} \circ \mathcal{P}^{(k)})_{(i)}\|_* = \sum_{i=1}^3 \alpha_i\|P_i\mathcal{A}_{(i)}Q_i\|_* = \sum_{i=1}^3 \alpha_i\|\mathcal{A}_{(i)}\|_*$. □

**Theorem 3.** *$\mathcal{S}_\tau(\mathcal{Y}) = \mathcal{S}_\tau(\mathcal{Y}\circ\mathcal{P}^{(k)})\circ(\mathcal{P}^{(k)})^{-1}(k = 1, 2, 3)$, where $\mathcal{S}_\tau(\mathcal{Y}) = \arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{Y}-\mathcal{X}\|_F^2 + \tau\sum_{i=1}^3 \alpha_i\|\mathcal{X}_{(i)}\|_*$.*

*Proof.*

$$\begin{aligned} &\mathcal{S}_\tau(\mathcal{Y} \circ \mathcal{P}^{(k)}) \circ (\mathcal{P}^{(k)})^{-1} \\ =&(\arg\min_{\mathcal{Z}} \frac{1}{2}\|\mathcal{Y} \circ \mathcal{P}^{(k)} - \mathcal{Z}\|_F^2 + \tau\sum_{i=1}^3 \alpha_i\|\mathcal{Z}_{(i)}\|_*) \circ (\mathcal{P}^{(k)})^{-1} \\ =&\arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{Y} \circ \mathcal{P}^{(k)} - \mathcal{X} \circ \mathcal{P}^{(k)}\|_F^2 + \tau\sum_{i=1}^3 \alpha_i\|(\mathcal{X} \circ \mathcal{P}^{(k)})_{(i)}\|_* \\ =&\arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{Y} - \mathcal{X}\|_F^2 + \tau\sum_{i=1}^3 \alpha_i\|\mathcal{X}_{(i)}\|_*, \end{aligned} \tag{2}$$

where the second equation holds by letting $\mathcal{X} = \mathcal{Z} \circ (\mathcal{P}^{(k)})^{-1}$, and the third equation holds by the property of $\mathcal{P}^{(k)}$ and Property 2. □

**Property 3.** *(Horizontal SPI of tensor nuclear norm) Tensor nuclear norm satisfy HSPI (Horizontal SPI),* i.e. *$\|\mathcal{A}\|_* = \|\mathcal{A} \circ \mathcal{P}^{(1)}\|_*$, for any horizontal slice permutations $\mathcal{P}^{(1)}$.*

*Proof.* By the definition of $\mathrm{bcirc}(\mathcal{A})$, exist two permutation matrices $P$ and $Q$ such that $\mathrm{bcirc}(\mathcal{A}\circ\mathcal{P}^{(1)}) = P \cdot \mathrm{bcirc}(\mathcal{A}) \cdot Q$. Therefore, $\|\mathcal{A}\circ\mathcal{P}^{(1)}\|_* = \|\mathcal{A}\circ\mathcal{P}^{(1)}\|_{a,*} = \frac{1}{n_3}\|\mathrm{bcirc}(\mathcal{A}\circ\mathcal{P}^{(1)})\|_* = \frac{1}{n_3}\|P \cdot \mathrm{bcirc}(\mathcal{A}) \cdot Q\|_*$. By Property 1, $\frac{1}{n_3}\|P \cdot \mathrm{bcirc}(\mathcal{A}) \cdot Q\|_* = \frac{1}{n_3}\|\mathrm{bcirc}(\mathcal{A})\|_* = \|\mathcal{A}\|_{a,*} = \|\mathcal{A}\|_*$. Thus $\|\mathcal{A} \circ \mathcal{P}^{(1)}\|_* = \|\mathcal{A}\|_*$. □

**Property 4.** *(Lateral SPI of tensor nuclear norm) tensor nuclear norm satisfy LSPI (Lateral SPI),* i.e. *$\|\mathcal{A}\|_* = \|\mathcal{A} \circ \mathcal{P}^{(2)}\|_*$, for any lateral slices permutations $\mathcal{P}^{(2)}$.*

*Proof.* Similar to the proof of Property 3. □

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**Property 5.** *For same circle* $\boldsymbol{C}^1 = \{i_1, i_2, ..., i_{n_3}, i_1\}$ *and* $\boldsymbol{C}^2 = \{i_k, i_{k+1}, ..., i_{n_3}, ..., i_{k-1}, i_k\}$,

$$\|\mathcal{A} \circ \mathcal{P}^{(3)}_{\boldsymbol{Or}^1}\|_* = \|\mathcal{A} \circ \mathcal{P}^{(3)}_{\boldsymbol{Or}^2}\|_*,$$

*where* $\boldsymbol{Or}^1 = \{i_1, i_2, ..., i_{n_3}\}$ *is obtained by* $\boldsymbol{C}^1$, *and* $\boldsymbol{Or}^2 = \{i_k, i_{k+1}, ..., i_{n_3}, ..., i_{k-1}\}$ *is obtained by* $\boldsymbol{C}^2$.

*Proof.*

$$\begin{aligned}
&\mathrm{bcirc}(\mathcal{A} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1}) \\
=&\begin{pmatrix} \mathcal{A}_{:,:,i_1} & \mathcal{A}_{:,:,i_{n_3}} & \cdots & \mathcal{A}_{:,:,i_3} & \mathcal{A}_{:,:,i_2} \\ \mathcal{A}_{:,:,i_2} & \mathcal{A}_{:,:,i_1} & \cdots & \mathcal{A}_{:,:,i_4} & \mathcal{A}_{:,:,i_3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{A}_{:,:,i_{n_3-1}} & \mathcal{A}_{:,:,i_{n_3-2}} & \cdots & \mathcal{A}_{:,:,i_1} & \mathcal{A}_{:,:,i_{n_3}} \\ \mathcal{A}_{:,:,i_{n_3}} & \mathcal{A}_{:,:,i_{n_3-1}} & \cdots & \mathcal{A}_{:,:,i_2} & \mathcal{A}_{:,:,i_1} \end{pmatrix} \\
\longrightarrow&\begin{pmatrix} \mathcal{A}_{:,:,i_k} & \mathcal{A}_{:,:,i_{k-1}} & \cdots & \mathcal{A}_{:,:,i_{k+2}} & \mathcal{A}_{:,:,i_{k+1}} \\ \mathcal{A}_{:,:,i_{k+1}} & \mathcal{A}_{:,:,i_k} & \cdots & \mathcal{A}_{:,:,i_{k+3}} & \mathcal{A}_{:,:,i_{k+2}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{A}_{:,:,i_{k-2}} & \mathcal{A}_{:,:,i_{k-3}} & \cdots & \mathcal{A}_{:,:,i_k} & \mathcal{A}_{:,:,i_{k-1}} \\ \mathcal{A}_{:,:,i_{k-1}} & \mathcal{A}_{:,:,i_{k-2}} & \cdots & \mathcal{A}_{:,:,i_{k+1}} & \mathcal{A}_{:,:,i_k} \end{pmatrix} \\
=&\,\mathrm{bcirc}(\mathcal{A} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^2}).
\end{aligned} \tag{3}$$

Therefore $\|\mathcal{A} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1}\|_* = \|\mathcal{A} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1}\|_{a,*} = \frac{1}{n_3}\|\mathrm{bcirc}(\mathcal{A} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1})\|_* = \frac{1}{n_3}\|\mathrm{bcirc}(\mathcal{A} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^2})\|_* = \|\mathcal{A} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^2}\|_{a,*} = \|\mathcal{A} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^2}\|_*$. □

**Theorem 4.** *For same circle* $\boldsymbol{C}^1 = \{i_1, i_2, ..., i_{n_3}, i_1\}$ *and* $\boldsymbol{C}^2 = \{i_k, i_{k+1}, ..., i_{n_3}, ..., i_{k-1}, i_k\}$,

$$\mathcal{D}_\tau(\mathcal{Y} \circ \mathcal{P}^{(3)}_{\boldsymbol{Or}^1}) \circ \mathcal{P}^{(3)^{-1}}_{\boldsymbol{Or}^1} = \mathcal{D}_\tau(\mathcal{Y} \circ \mathcal{P}^{(3)}_{\boldsymbol{Or}^2}) \circ \mathcal{P}^{(3)^{-1}}_{\boldsymbol{Or}^2} \tag{4}$$

*where* $\mathcal{D}_\tau(\mathcal{A}) = \arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{A} - \mathcal{X}\|_F^2 + \tau\|\mathcal{X}\|_*$, $\boldsymbol{Or}^1 = \{i_1, i_2, ..., i_{n_3}\}$ *is obtained by* $\boldsymbol{C}^1$, *and* $\boldsymbol{Or}^2 = \{i_k, i_{k+1}, ..., i_{n_3}, ..., i_{k-1}\}$ *is obtained by* $\boldsymbol{C}^2$.

*Proof.*

$$\begin{aligned}
&(\arg\min_{\mathcal{Z}} \frac{1}{2}\|\mathcal{Y} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1} - \mathcal{Z}\|_F^2 + \tau\|\mathcal{Z}\|_*) \circ \mathcal{P}^{(3)^{-1}}_{\mathbf{Or}^1} \\
=&\arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{Y} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1} - \mathcal{X} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1}\|_F^2 + \tau\|\mathcal{X} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1}\|_* \\
=&\arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{Y} - \mathcal{X}\|_F^2 + \tau\|\mathcal{X} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1}\|_*,
\end{aligned}$$

where the first equation holds by letting $\mathcal{X} = \mathcal{Z} \circ (\mathcal{P}^{(3)}_{\mathbf{Or}^1})^{-1}$.

By Property 5, $\|\mathcal{X} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1}\|_* = \|\mathcal{X} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^2}\|_*$. Therefore,

$$\begin{aligned}
&\arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{Y} - \mathcal{X}\|_F^2 + \tau\|\mathcal{X} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1}\|_* \\
=&\arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{Y} - \mathcal{X}\|_F^2 + \tau\|\mathcal{X} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^2}\|_* \\
=&\arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{Y} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^2} - \mathcal{X} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^2}\|_F^2 + \tau\|\mathcal{X} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^2}\|_* \\
=&(\arg\min_{\mathcal{Z}} \frac{1}{2}\|\mathcal{Y} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^2} - \mathcal{Z}\|_F^2 + \tau\|\mathcal{Z}\|_*) \circ \mathcal{P}^{(3)^{-1}}_{\mathbf{Or}^2},
\end{aligned}$$

where the third equation holds by letting $\mathcal{Z} = \mathcal{X} \circ (\mathcal{P}^{(3)}_{\mathbf{Or}^2})$. The conclusion holds. □

**Property 6.** *For* $\mathcal{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, *if* $n_3 \leq 3$, *then tensor nuclear norm satisfy frontal slice permutations invariance (FSPI),* i.e. $\|\mathcal{A}\|_* = \|\mathcal{A} \circ \mathcal{P}^{(3)}_{\boldsymbol{Or}}\|_*$ *for any frontal slice permutations* $\mathcal{P}^{(3)}_{\boldsymbol{Or}}$.

*Proof.* For $n_3 = 2$, let $\mathcal{B}_{:,:,1} = \mathcal{A}_{:,:,2}$ and $\mathcal{B}_{:,:,2} = \mathcal{A}_{:,:,1}$. Thus $\mathrm{bcirc}(\mathcal{B}) = \begin{pmatrix} \mathcal{B}_{:,:,1} & \mathcal{B}_{:,:,2} \\ \mathcal{B}_{:,:,2} & \mathcal{B}_{:,:,1} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_{:,:,2} & \mathcal{A}_{:,:,1} \\ \mathcal{A}_{:,:,1} & \mathcal{A}_{:,:,2} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{A}_{:,:,1} & \mathcal{A}_{:,:,2} \\ \mathcal{A}_{:,:,2} & \mathcal{A}_{:,:,1} \end{pmatrix} = \mathrm{bcirc}(\mathcal{A})$. Therefore, $\|\mathcal{A}\|_* = \|\mathcal{A}\|_{*,a} = \|\mathcal{B}\|_{*,a} = \|\mathcal{B}\|_*$.

For $n_3 = 3$, there is only one circle. Therefore, $\|\mathcal{A}\|_* = \|\mathcal{A} \circ \mathcal{P}^{(3)}_{\mathbf{Or}}\|_*$ by Property 5.

Thus, the conclusion holds.

□

**Theorem 5.** *For* $\mathcal{Y} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, *if* $n_3 \leq 3$, *then*

$$\mathcal{D}_\tau(\mathcal{Y}) = \mathcal{D}_\tau(\mathcal{Y} \circ \mathcal{P}^{(k)}) \circ \mathcal{P}^{(k)^{-1}} \tag{5}$$

*for* $k = 1, 2, 3$.

*Proof.*

$$\begin{aligned}
&\mathcal{D}_\tau(\mathcal{Y} \circ \mathcal{P}^{(k)}) \circ (\mathcal{P}^{(k)})^{-1} \\
=&(\arg\min_{\mathcal{Z}} \frac{1}{2}\|\mathcal{Y} \circ \mathcal{P}^{(k)} - \mathcal{Z}\|_F^2 + \tau\|\mathcal{Z}\|_*) \circ (\mathcal{P}^{(k)})^{-1} \\
=&\arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{Y} \circ \mathcal{P}^{(k)} - \mathcal{X} \circ \mathcal{P}^{(k)}\|_F^2 + \tau\|\mathcal{X} \circ \mathcal{P}^{(k)}\|_* \\
=&\arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{Y} - \mathcal{X}\|_F^2 + \tau\|\mathcal{X}\|_*,
\end{aligned} \tag{6}$$

where the second equation holds by letting $\mathcal{X} = \mathcal{Z} \circ (\mathcal{P}^{(k)})^{-1}$, and the third equation holds by the property of $\mathcal{P}^{(k)}$ and Property 3, 4 and 6. □

# References


Zhang, X.-D. 2017. *Matrix analysis and applications*. Cambridge University Press.