Handling Slice Permutations Variability in Tensor Recovery

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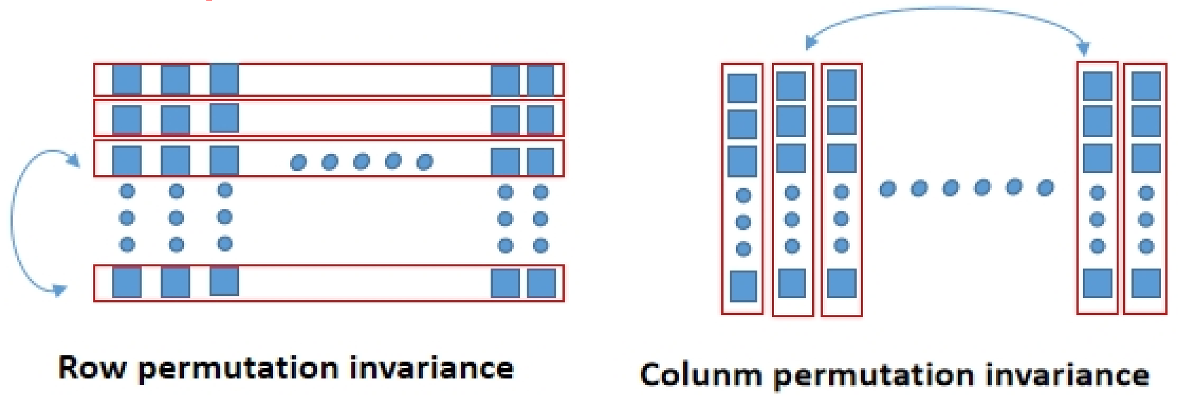
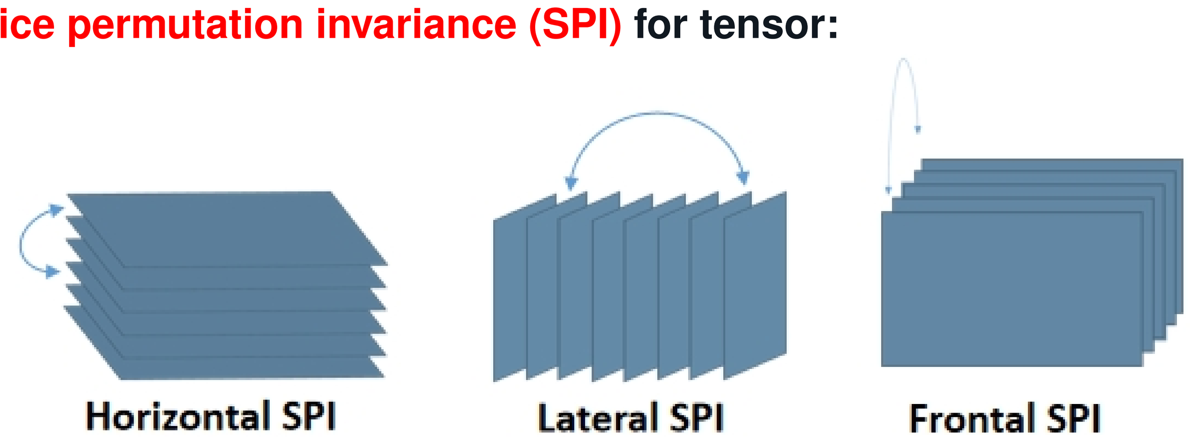
Motivation

► **An assumption for matrix recovery and tensor recovery**

- **Row/column permutation invariance for matrix:**



- **Slice permutation invariance (SPI) for tensor:**



► **A counter-example for tensor SPI: a huge gap between the tensor with different slices order**

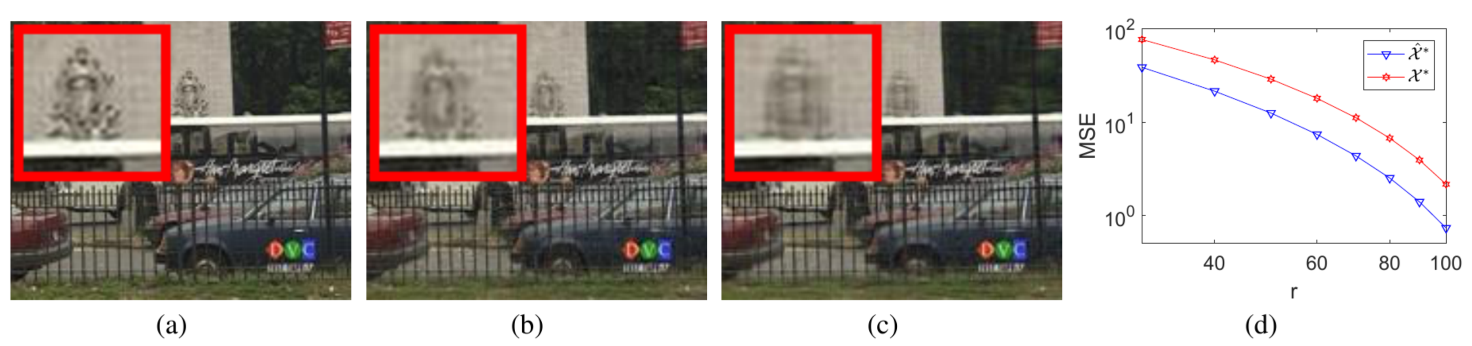


Figure 1: Color video ('bus') (modeled as a tensor $\mathcal{Y} \in \mathbb{R}^{144\times176\times90}$) can be approximated by low tubal rank tensor. Here, only first frame of visual results in (a)-(b) are presented. (a) The first frame of original video (b) approximation by tensor $\mathcal{X}^* \in \mathbb{R}^{144\times176\times90}$ with tubal rank $r = 30$. (MPSNR=32.45dB) (c) approximation by tensor $\hat{\mathcal{X}}^* \in \mathbb{R}^{144\times176\times90}$ with tubal rank $r = 30$. (MPSNR=29.27dB) (d) MSE results of $\mathcal{X}^*$ and $\hat{\mathcal{X}}^*$ comparison for different r.

SPI of Tensor Nuclear Norm

Theorem 1. *For same circle* $\mathbf{C}^1 = \{i_1, i_2, ..., i_{n_3}, i_1\}$ *and* $\mathbf{C}^2 = \{i_k, i_{k+1}, ..., i_{n_3}, ..., i_{k-1}, i_k\}$,

$$\mathcal{D}_\tau(\mathcal{Y} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^1}) \circ \mathcal{P}^{(3)^{-1}}_{\mathbf{Or}^1} = \mathcal{D}_\tau(\mathcal{Y} \circ \mathcal{P}^{(3)}_{\mathbf{Or}^2}) \circ \mathcal{P}^{(3)^{-1}}_{\mathbf{Or}^2} \quad (1)$$

where $\mathcal{D}_\tau(\mathcal{A}) = \arg\min_{\mathcal{X}} \frac{1}{2}\|\mathcal{A} - \mathcal{X}\|_F^2 + \tau\|\mathcal{X}\|_*$, $\mathbf{Or}^1 = \{i_1, i_2, ..., i_{n_3}\}$ *is obtained by* $\mathbf{C}^1$, *and* $\mathbf{Or}^2 = \{i_k, i_{k+1}, ..., i_{n_3}, ..., i_{k-1}\}$ *is obtained by* $\mathbf{C}^2$.

► **The SPI of tensor recovery for color image ($n_3 = 3$)**

Theorem 2. *For* $\mathcal{Y} \in \mathbb{R}^{n_1\times n_2\times n_3}$, *if* $n_3 \le 3$, *then*

$$\mathcal{D}_\tau(\mathcal{Y}) = \mathcal{D}_\tau(\mathcal{Y} \circ \mathcal{P}^{(k)}) \circ \mathcal{P}^{(k)^{-1}} \quad (2)$$

for $k = 1, 2, 3$.

Methodology

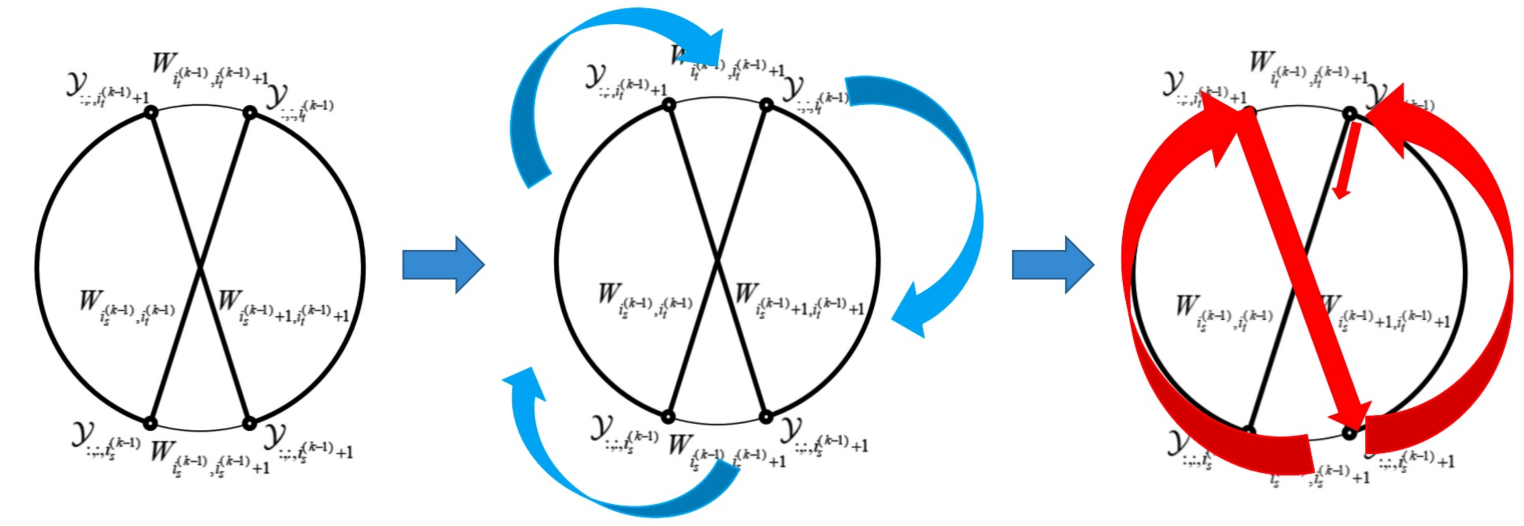
$$\min_{\mathcal{X}, \mathcal{P}^{(3)}_{\mathbf{Or}}} \frac{1}{2}\|\mathcal{Y} \circ \mathcal{P}^{(3)}_{\mathbf{Or}} - \mathcal{X}\|_F^2 + \tau\|\mathcal{X}\|_{*,a} \quad (3)$$

$$= \min_{\mathcal{X}, \mathcal{P}^{(3)}_{\mathbf{Or}}} \frac{1}{2n_3}\|\mathrm{bcirc}(\mathcal{Y} \circ \mathcal{P}^{(3)}_{\mathbf{Or}}) - \mathrm{bcirc}(\mathcal{X})\|_F^2 + \tau\,\|\mathrm{bcirc}(\mathcal{X})\|_*. \quad (4)$$

$$\begin{pmatrix} \mathcal{Y}_{:,:,\mathbf{Or}(1)} & \mathcal{Y}_{:,:,\mathbf{Or}(n_3)} & \cdots & \mathcal{Y}_{:,:,\mathbf{Or}(2)} \\ \mathcal{Y}_{:,:,\mathbf{Or}(2)} & \mathcal{Y}_{:,:,\mathbf{Or}(1)} & \cdots & \mathcal{Y}_{:,:,\mathbf{Or}(3)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Y}_{:,:,\mathbf{Or}(n_3)} & \mathcal{Y}_{:,:,\mathbf{Or}(n_3-1)} & \cdots & \mathcal{Y}_{:,:,\mathbf{Or}(1)} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathcal{Y}_{:,:,\mathbf{Or}(1)} & \mathcal{Y}_{:,:,\mathbf{Or}(2)} & \cdots & \mathcal{Y}_{:,:,\mathbf{Or}(n_3)} \\ \mathcal{Y}_{:,:,\mathbf{Or}(2)} & \mathcal{Y}_{:,:,\mathbf{Or}(3)} & \cdots & \mathcal{Y}_{:,:,\mathbf{Or}(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Y}_{:,:,\mathbf{Or}(n_3)} & \mathcal{Y}_{:,:,\mathbf{Or}(1)} & \cdots & \mathcal{Y}_{:,:,\mathbf{Or}(n_3-1)} \end{pmatrix}. \quad (5)$$

► $\mathcal{Y}$ **with similar adjacent frontal slices can be approximated by a lower rank matrix.**

► **The key point to find a better order sequence of the frontal slice is to solve a Minimum Hamiltonian circle problem.**



► **From [1], the simplest idea for getting a Minimum Hamiltonian circle is that, when we get Circle k, we can make appropriate modifications for circle k to get another circle $k+1$ with a smaller weight.**

Algorithm 2: Tensor recovery for SPV (TRSPV)

Input: $\mathcal{Y} \in \mathbb{R}^{n_1\times n_2\times n_3}$, and $Iternum$.
Output: $\mathbf{C}^*(\mathcal{Y})$ and $\mathcal{T}_\tau(\mathcal{Y})$
Compute weight matrix W;
Initialize circle $\mathbf{C}^{(0)} = \{i_1^{(0)}, i_2^{(0)}, ..., i_{n_3}^{(0)}, i_1^{(0)}\}$, and $k = 0$;
while $k \le Iternum$ **do**
  $k = k + 1$;
  if *there are different* $i_s^{(k-1)}, i_t^{(k-1)}, i_s^{(k-1)} + 1, i_t^{(k-1)} + 1$ *in* $\mathbf{C}^{(k-1)}$ *which make* $W_{i_s^{(k-1)}, i_t^{(k-1)}}(\mathcal{Y}) + W_{i_s^{(k-1)}+1, i_t^{(k-1)}+1}(\mathcal{Y}) < W_{i_s^{(k-1)}, i_s^{(k-1)}+1}(\mathcal{Y}) + W_{i_t^{(k-1)}, i_t^{(k-1)}+1}(\mathcal{Y})$ **then**
    $\mathbf{C}^{(k)} = \{i_t^{(k-1)}, i_s^{(k-1)}\} \bigcup \mathbf{C}^{(k-1)^{-1}}(i_{t+1}^{(k-1)}, i_s^{(k-1)}) \bigcup \{i_{t+1}^{(k-1)}, i_{s+1}^{(k-1)}\} \bigcup \mathbf{C}^{(k-1)}(i_{s+1}^{(k-1)}, i_t^{(k-1)})$;
  else
    $\mathbf{C}^{(k)} = \mathbf{C}^{(k-1)}$;
    break;
  end
end
Obtain $\mathbf{C}^*(\mathcal{Y}) = \mathbf{C}^{(k)}$, and compute $\mathcal{T}_\tau(\mathcal{Y}) = \mathcal{D}_\tau(\mathcal{Y}^{\mathbf{Or}^*})$, where $\mathbf{Or}^*$ obtained by $\mathbf{C}^*(\mathcal{Y})$;

Algorithm 3: TRPCA for SPV (TRPCA-SPV)

Initialize: $\mathcal{L}^{(0)} = \mathcal{S}^{(0)} = \mathcal{Q}^{(0)} = \mathcal{Y}^{(0)} = \mathbf{0}$, $\rho = 1.1$, $\mu_0 = 1e-3$, $\epsilon = 1e-8$, $\kappa > 0$.
while *not converged* **do**
  1. Update $\mathbf{Or}^*$ by
  If $\kappa = 1$ or $k \mod \kappa = 1$, update $\mathbf{Or}^*$ by $\mathbf{C}^*(\mathcal{M}^{(k)})$, where $\mathcal{M}^{(k)} = \mathcal{P} - \mathcal{S}^{(k)} - \frac{\mathcal{Q}^{(k)}}{\mu_k}$;
  2.Update $\mathcal{L}^{(k+1)}$ by $\mathcal{L}^{(k+1)} = \arg\min_{\mathcal{L}} \|\mathcal{L}\|_* + \frac{\mu_k}{2}\|\mathcal{L} - (\mathcal{M}^{(k)})^{\mathbf{Or}^*}\|_F^2$;
  3. Update $\mathcal{S}^{(k+1)}$ by $\mathcal{S}^{(k+1)} = \arg\min_{\mathcal{S}} \lambda\|\mathcal{S}^{\mathbf{Or}^*}\|_1 + \frac{\mu_k}{2}\|\mathcal{L}^{(k+1)} + \mathcal{S}^{\mathbf{Or}^*} - \mathcal{P}^{\mathbf{Or}^*} + (\frac{\mathcal{Q}^{(k)}}{\mu_k})^{\mathbf{Or}^*}\|_F^2$;
  4.$(\mathcal{Q}^{(k+1)})^{\mathbf{Or}^*} = (\mathcal{Q}^{(k)})^{\mathbf{Or}^*} + \mu(\mathcal{L}^{(k+1)} + (\mathcal{S}^{(k+1)})^{\mathbf{Or}^*} - \mathcal{P}^{\mathbf{Or}^*})$;
  5.Update μ_{k+1} by $\mu_{k+1} = \min(\rho\mu_k, \mu_{\max})$;
  6. Check the convergence conditions
  $\|\mathcal{L}^{(k+1)} - \mathcal{L}^{(k)}\|_\infty \le \epsilon$,
  $\|(\mathcal{S}^{(k+1)})^{\mathbf{Or}^*} - (\mathcal{S}^{(k)})^{\mathbf{Or}^*}\|_\infty \le \epsilon$,
  $\|\mathcal{L}^{(k+1)} + (\mathcal{S}^{(k+1)})^{\mathbf{Or}^*} - \mathcal{P}^{\mathbf{Or}^*}\|_\infty \le \epsilon$;
end

Experiments and Results

► **Experiment 1: Image classification**

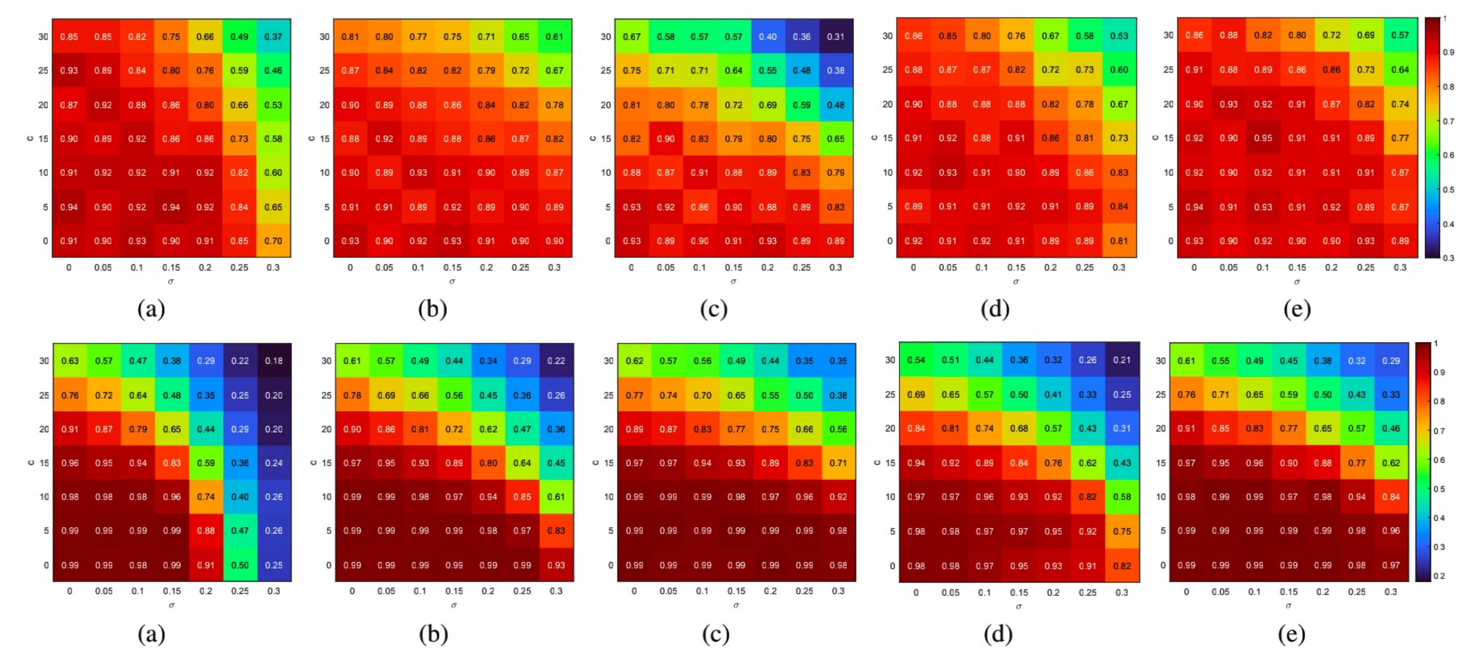


Figure 2: Classification accuracies of the 5 algorithms on ORL database and CMU PIE database: (a) RPCA[3] (b) SNN[4] (c) Liu[2] (d) TRPCA[5] (e) TRPCA-SPV

► **Experiment 2: Image sequence recovery**

		Botswana					Pavia University				
δ	c	RPCA	SNN	Liu	TRPCA	TRPCA-SPV	RPCA	SNN	Liu	TRPCA	TRPCA-SPV
5	5%	29.90	34.52	36.82	32.06	**38.44**	27.56	29.82	32.03	30.65	**36.60**
	15%	29.04	33.02	35.34	30.06	**37.11**	26.90	29.21	31.60	28.07	**35.39**
	25%	27.73	30.81	32.92	28.78	**34.98**	25.55	27.96	30.53	26.03	**33.48**
15	5%	28.11	30.91	32.42	31.11	**34.21**	25.58	27.19	28.07	30.22	**31.51**
	15%	27.32	29.47	30.92	28.99	**32.34**	24.77	26.43	28.35	27.17	**30.38**
	25%	25.78	27.23	28.48	27.18	**29.67**	23.20	24.96	26.99	24.67	**27.76**
25	5%	26.84	29.17	30.37	29.34	**31.65**	23.63	25.12	26.94	28.50	**29.02**
	15%	26.05	27.55	28.67	26.83	**29.77**	22.74	24.30	26.30	25.21	**27.49**
	25%	24.29	25.14	26.06	24.18	**26.79**	21.11	22.76	24.77	22.49	**24.81**

► **Experiment 3: Sensitivity analysis of parameters**

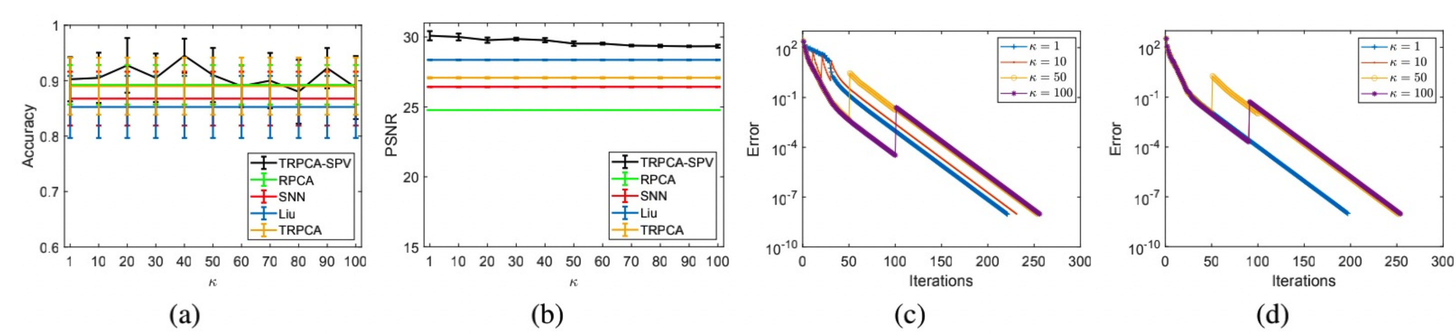


Figure 3: Sensitivity analysis of parameter κ for TRPCA-SPV on (a) ORL database and (b) Pavia University; Convergence analysis for TRPCA-SPV with different κ on (c) ORL database and (d) Pavia University.

References

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