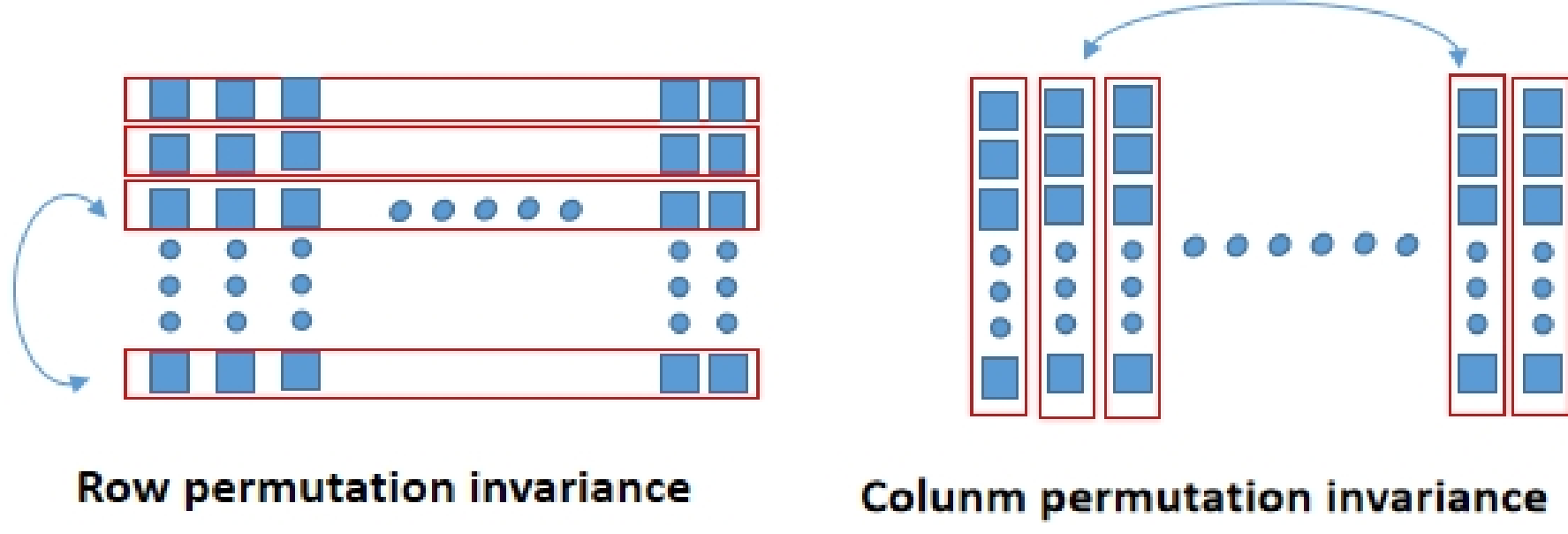




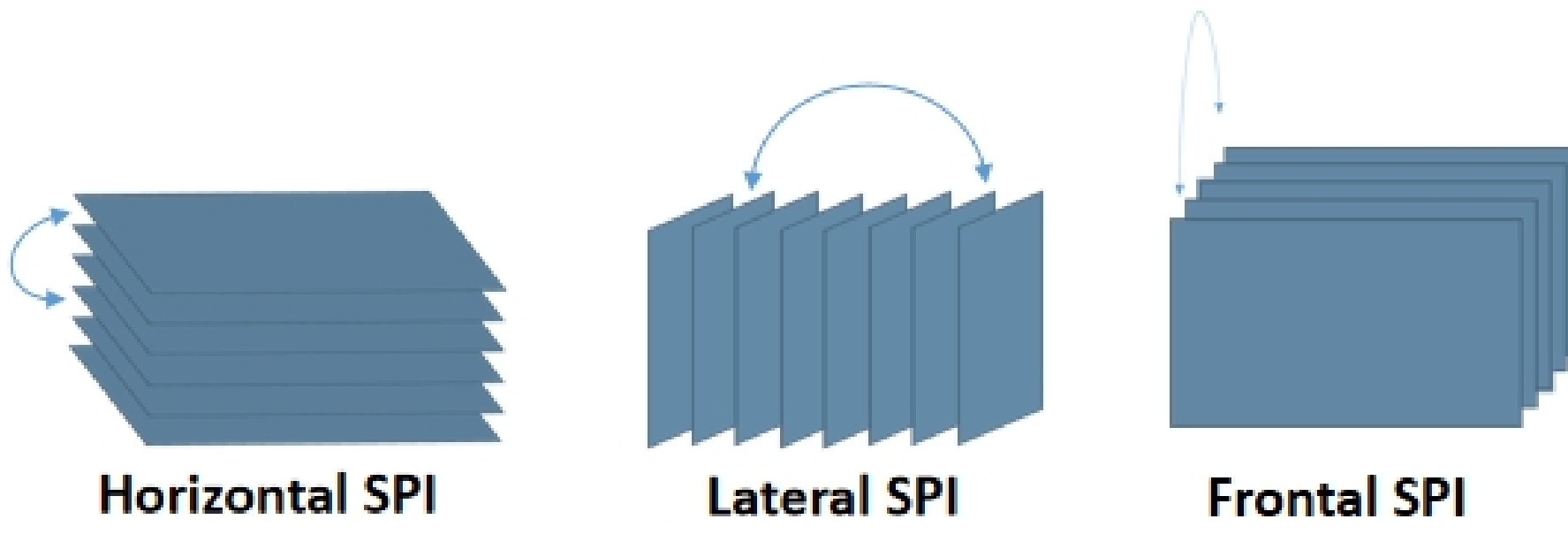
Motivation

► An assumption for matrix recovery and tensor recovery

• Row/column permutation invariance for matrix:



• Slice permutation invariance (SPI) for tensor:



► A counter-example for tensor SPI: a huge gap between the tensor with different slices order

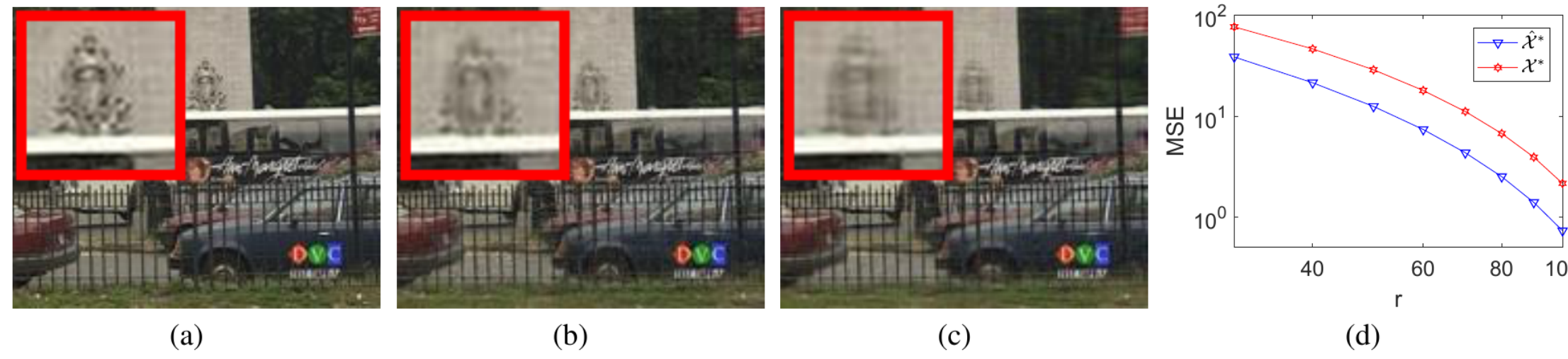


Figure 1: Color video "bus" (modeled as a tensor $\mathcal{Y} \in \mathbb{R}^{144 \times 176 \times 90}$) can be approximated by low tubal rank tensor. Here, only first frame of visual results in (a)-(b) are presented. (a) The first frame of original video (b) approximation by tensor $\mathcal{X}^* \in \mathbb{R}^{144 \times 176 \times 90}$ with tubal rank $r = 30$. (MPSNR=32.45dB) (c) approximation by tensor $\mathcal{X} \in \mathbb{R}^{144 \times 176 \times 90}$ with tubal rank $r = 30$. (MPSNR=29.27dB) (d) MSE results of \mathcal{X}^* and \mathcal{X} comparison for different r .

SPI of Tensor Nuclear Norm

Theorem 1. For same circle $\mathcal{C}^1 = \{i_1, i_2, \dots, i_{n_3}, i_1\}$ and $\mathcal{C}^2 = \{i_k, i_{k+1}, \dots, i_{n_3}, \dots, i_{k-1}, i_k\}$,

$$\mathcal{D}_\tau(\mathcal{Y} \circ \mathcal{P}_{\mathbf{Or}^1}^{(3)}) \circ \mathcal{P}_{\mathbf{Or}^1}^{(3)-1} = \mathcal{D}_\tau(\mathcal{Y} \circ \mathcal{P}_{\mathbf{Or}^2}^{(3)}) \circ \mathcal{P}_{\mathbf{Or}^2}^{(3)-1} \quad (1)$$

where $\mathcal{D}_\tau(\mathcal{A}) = \arg \min_{\mathcal{X}} \frac{1}{2} \|\mathcal{A} - \mathcal{X}\|_F^2 + \tau \|\mathcal{X}\|_*$, $\mathbf{Or}^1 = \{i_1, i_2, \dots, i_{n_3}\}$ is obtained by \mathcal{C}^1 , and $\mathbf{Or}^2 = \{i_k, i_{k+1}, \dots, i_{n_3}, \dots, i_{k-1}\}$ is obtained by \mathcal{C}^2 .

► The SPI of tensor recovery for color image ($n_3 = 3$)

Theorem 2. For $\mathcal{Y} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, if $n_3 \leq 3$, then

$$\mathcal{D}_\tau(\mathcal{Y}) = \mathcal{D}_\tau(\mathcal{Y} \circ \mathcal{P}^{(k)}) \circ \mathcal{P}^{(k)-1} \quad (2)$$

for $k = 1, 2, 3$.

Methodology

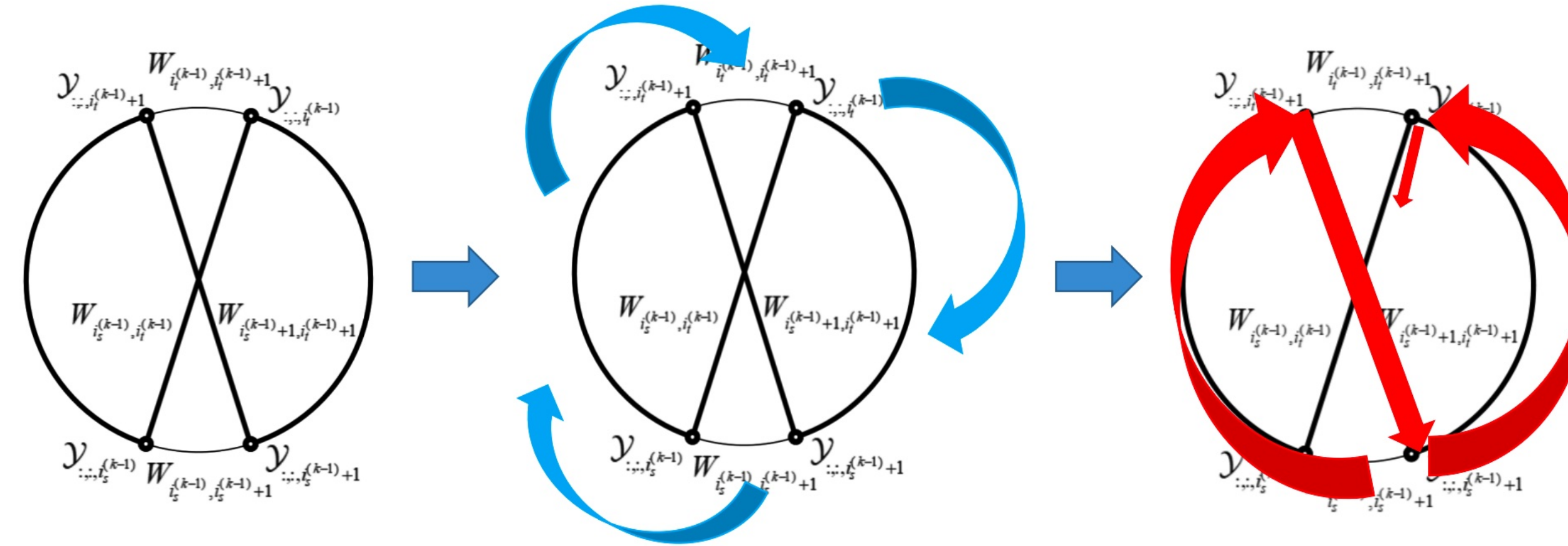
$$\min_{\mathcal{X}, \mathcal{P}_{\mathbf{Or}}^{(3)}} \frac{1}{2} \|\mathcal{Y} \circ \mathcal{P}_{\mathbf{Or}}^{(3)} - \mathcal{X}\|_F^2 + \tau \|\mathcal{X}\|_{*,a} \quad (3)$$

$$= \min_{\mathcal{X}, \mathcal{P}_{\mathbf{Or}}^{(3)}} \frac{1}{2n_3} \|\text{bcirc}(\mathcal{Y} \circ \mathcal{P}_{\mathbf{Or}}^{(3)}) - \text{bcirc}(\mathcal{X})\|_F^2 + \tau \|\text{bcirc}(\mathcal{X})\|_* \quad (4)$$

$$\begin{pmatrix} \mathcal{Y}_{:, :, \mathbf{Or}(1)} & \mathcal{Y}_{:, :, \mathbf{Or}(n_3)} & \cdots & \mathcal{Y}_{:, :, \mathbf{Or}(2)} \\ \mathcal{Y}_{:, :, \mathbf{Or}(2)} & \mathcal{Y}_{:, :, \mathbf{Or}(1)} & \cdots & \mathcal{Y}_{:, :, \mathbf{Or}(3)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Y}_{:, :, \mathbf{Or}(n_3)} & \mathcal{Y}_{:, :, \mathbf{Or}(n_3-1)} & \cdots & \mathcal{Y}_{:, :, \mathbf{Or}(1)} \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{Y}_{:, :, \mathbf{Or}(1)} & \mathcal{Y}_{:, :, \mathbf{Or}(2)} & \cdots & \mathcal{Y}_{:, :, \mathbf{Or}(n_3)} \\ \mathcal{Y}_{:, :, \mathbf{Or}(2)} & \mathcal{Y}_{:, :, \mathbf{Or}(3)} & \cdots & \mathcal{Y}_{:, :, \mathbf{Or}(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Y}_{:, :, \mathbf{Or}(n_3)} & \mathcal{Y}_{:, :, \mathbf{Or}(1)} & \cdots & \mathcal{Y}_{:, :, \mathbf{Or}(n_3-1)} \end{pmatrix} \quad (5)$$

► \mathcal{Y} with similar adjacent frontal slices can be approximated by a lower rank matrix.

► The key point to find a better order sequence of the frontal slice is to solve a Minimum Hamiltonian circle problem.



► From [1], the simplest idea for getting a Minimum Hamiltonian circle is that, when we get Circle k, we can make appropriate modifications for circle k to get another circle k + 1 with a smaller weight.

Algorithm 2: Tensor recovery for SPV (TRSPV)

Input: $\mathcal{Y} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, and Iternum.
Output: \mathbf{C}^* (\mathcal{Y}) and $\mathcal{T}_\tau(\mathcal{Y})$
 Compute weight matrix W ;
 Initialize circle $\mathbf{C}^{(0)} = \{i_1^{(0)}, i_2^{(0)}, \dots, i_{n_3}^{(0)}, i_1^{(0)}\}$, and $k = 0$;
while $k \leq \text{Iternum}$ **do**
 $k = k + 1$;
 if there are different $i_s^{(k-1)}, i_t^{(k-1)}, i_{s+1}^{(k-1)} + 1, i_t^{(k-1)} + 1$ in $\mathbf{C}^{(k-1)}$ which make $W_{i_s^{(k-1)}, i_t^{(k-1)}}(\mathcal{Y}) + W_{i_t^{(k-1)}+1, i_{s+1}^{(k-1)}+1}(\mathcal{Y}) < W_{i_s^{(k-1)}+1, i_t^{(k-1)}+1}(\mathcal{Y}) + W_{i_t^{(k-1)}, i_{s+1}^{(k-1)}}(\mathcal{Y})$ **then**
 $\mathbf{C}^{(k)} = \{i_t^{(k-1)}, i_s^{(k-1)}\} \cup \mathbf{C}^{(k-1)-1} \setminus \{i_{t+1}^{(k-1)}, i_{s+1}^{(k-1)}\} \cup \mathbf{C}^{(k-1)} \setminus \{i_{s+1}^{(k-1)}, i_t^{(k-1)}\}$;
 else
 $\mathbf{C}^{(k)} = \mathbf{C}^{(k-1)}$;
 break;
 end
end
 Obtain $\mathbf{C}^*(\mathcal{Y}) = \mathbf{C}^{(k)}$, and compute $\mathcal{T}_\tau(\mathcal{Y}) = \mathcal{D}_\tau(\mathcal{Y} \circ \mathbf{Or}^*)$, where \mathbf{Or}^* obtained by $\mathbf{C}^*(\mathcal{Y})$;

Algorithm 3: TRPCA for SPV (TRPCA-SPV)

Initialize: $\mathcal{L}^{(0)} = \mathcal{S}^{(0)} = \mathcal{Q}^{(0)} = \mathcal{Y}^{(0)} = \mathbf{0}$, $\rho = 1.1$, $\mu_0 = 1e - 3$, $\epsilon = 1e - 8$, $\kappa > 0$.
while not converged do
 1. Update \mathbf{Or}^* by
 If $\kappa = 1$ or $\kappa \bmod \kappa = 1$, update \mathbf{Or}^* by $\mathbf{C}^*(\mathcal{M}^{(k)})$, where $\mathcal{M}^{(k)} = \mathcal{P} - \mathcal{S}^{(k)} - \frac{\mathcal{Q}^{(k)}}{\mu_k}$;
 2. Update $\mathcal{L}^{(k+1)}$ by $\mathcal{L}^{(k+1)} = \arg \min_{\mathcal{L}} \|\mathcal{L}\|_* + \frac{\mu_k}{2} \|\mathcal{L} - (\mathcal{M}^{(k)}) \mathbf{Or}^*\|_F^2$;
 3. Update $\mathcal{S}^{(k+1)}$ by $\mathcal{S}^{(k+1)} = \arg \min_{\mathcal{S}} \lambda \|\mathcal{S} \mathbf{Or}^*\|_1 + \frac{\mu_k}{2} \|\mathcal{L}^{(k+1)} + \mathcal{S} \mathbf{Or}^* - \mathcal{P} \mathbf{Or}^* + (\frac{\mathcal{Q}^{(k)}}{\mu_k}) \mathbf{Or}^*\|_F^2$;
 4. $(\mathcal{Q}^{(k+1)}) \mathbf{Or}^* = (\mathcal{Q}^{(k)}) \mathbf{Or}^* + \mu(\mathcal{L}^{(k+1)} + (\mathcal{S}^{(k+1)}) \mathbf{Or}^* - \mathcal{P} \mathbf{Or}^*)$;
 5. Update μ_{k+1} by $\mu_{k+1} = \min(\rho \mu_k, \mu_{\max})$;
 6. Check the convergence conditions
 $\|\mathcal{L}^{(k+1)} - \mathcal{L}^{(k)}\|_\infty \leq \epsilon$,
 $\|(\mathcal{S}^{(k+1)}) \mathbf{Or}^* - (\mathcal{S}^{(k)}) \mathbf{Or}^*\|_\infty \leq \epsilon$,
 $\|\mathcal{L}^{(k+1)} + (\mathcal{S}^{(k+1)}) \mathbf{Or}^* - \mathcal{P} \mathbf{Or}^*\|_\infty \leq \epsilon$;
end

Experiments and Results

► Experiment 1: Image classification

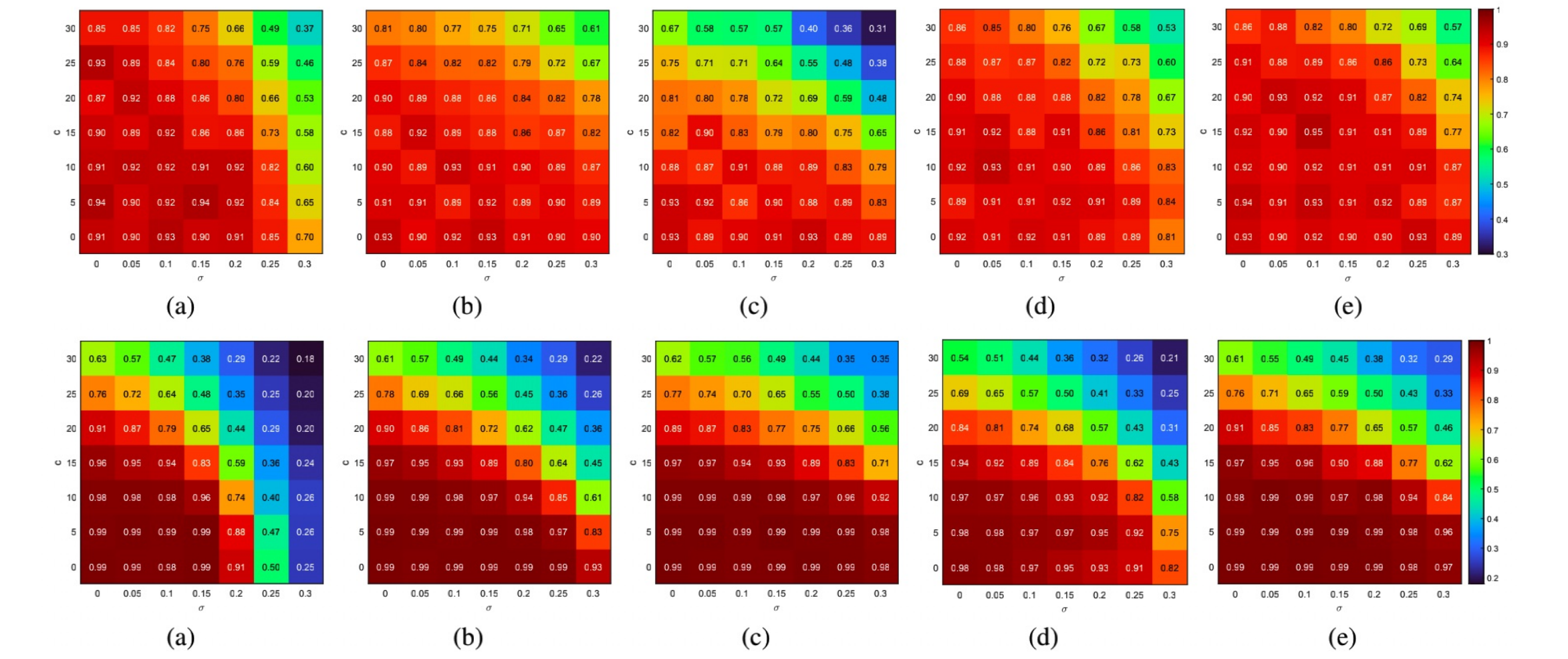


Figure 2: Classification accuracies of the 5 algorithms on ORL database and CMU PIE database: (a) RPCA[3] (b) SNN[4] (c) Liu[2] (d) TRPCA[5] (e) TRPCA-SPV

► Experiment 2: Image sequence recovery

δ	c	Botswana					Pavia University				
		RPCA	SNN	Liu	TRPCA	TRPCA-SPV	RPCA	SNN	Liu	TRPCA	TRPCA-SPV
5	5%	29.90	34.52	36.82	32.06	38.44	27.56	29.82	32.03	30.65	36.60
	15%	29.04	33.02	35.34	30.06	37.11	26.90	29.21	31.60	28.07	35.39
	25%	27.73	30.81	32.92	28.78	34.98	25.55	27.96	30.53	26.03	33.48
15	5%	28.11	30.91	32.42	31.11	34.21	25.58	27.19	28.07	30.22	31.51
	15%	27.32	29.47	30.92	28.99	32.34	24.77	26.43	28.35	27.17	30.38
	25%	25.78	27.23	28.48	27.18	29.67	23.20	24.96	26.99	24.67	27.76
25	5%	26.84	29.17	30.37	29.34	31.65	23.63	25.12	26.94	28.50	29.02
	15%	26.05	27.55	28.67	26.83	29.77	22.74	24.30	26.30	25.21	27.49
	25%	24.29	25.14	26.06	24.18	26.79	21.11	22.76	24.77	22.49	24.81

► Experiment 3: Sensitivity analysis of parameters

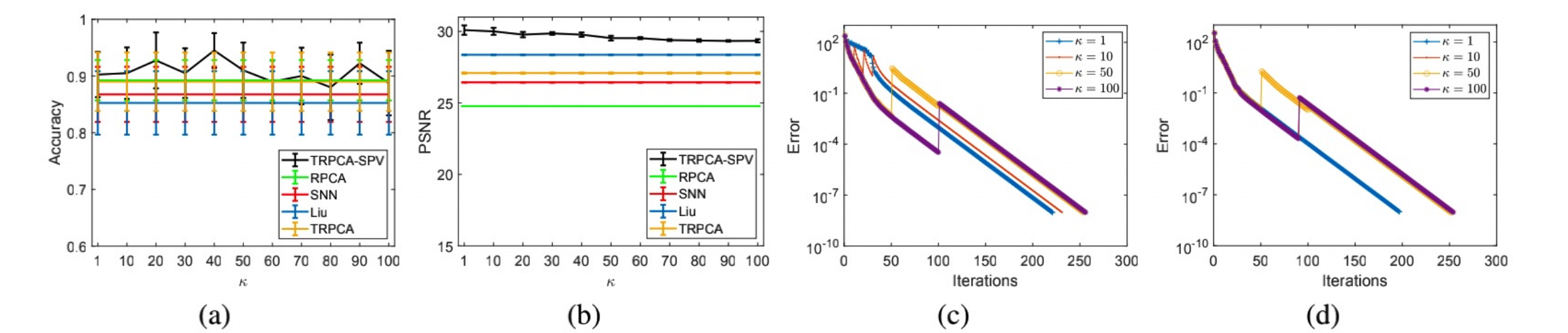


Figure 3: Sensitivity analysis of parameter κ for TRPCA-SPV on (a) ORL database and (b) Pavia University; Convergence analysis for TRPCA-SPV with different κ on (c) ORL database and (d) Pavia University.

References

- [1] John Adrian Bondy, Uppaluri Siva Ramachandra Murty, et al. *Graph theory with applications*. Vol. 290. Macmillan London, 1976.
- [2] Emmanuel J Candes and Yaniv Plan. "Matrix completion with noise". In: *Proceedings of the IEEE* 98.6 (2010), pp. 925–936.
- [3] Emmanuel J Candès et al. "Robust principal component analysis?" In: *Journal of the ACM (JACM)* 58.3 (2011), p. 1–37.
- [4] Silvia Gandy, Benjamin Recht, and Isao Yamada. "Tensor completion and low-n-rank tensor recovery via convex optimization". In: *Inverse Problems* 27.2 (2011), p. 025010.
- [5] Canyi Lu et al. "Tensor robust principal component analysis with a new tensor nuclear norm". In: *IEEE Transactions on Pattern Analysis and Machine Intelligence* 42.4 (2019), pp. 925–938.